Green functions and Euclidean fields near the bifurcate Killing horizon

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Abstract

We approximate a Euclidean version of a $D+1$ dimensional manifold with a bifurcate Killing horizon by a product of the two dimensional Rindler space $\mathcal{R}_2$ and a $D-1$ dimensional Riemannian manifold $\mathcal{M}$. We obtain approximate formulas for the Green functions. We study the behaviour of Green functions near the horizon and their dimensional reduction. We show that if $\mathcal{M}$ is compact then the massless minimally coupled quantum field contains a zero mode which is a conformal invariant free field on R^2. Then, the Green function near the horizon can be approximated by the Green function of the two-dimensional quantum field theory. The correction term is exponentially small away from the horizon. If the volume of a geodesic ball is growing to infinity with its radius then the Green function cannot be approximated by a two-dimensional one.

1 Introduction

We are interested in a study of the quantum version of the phenomena associated with a motion of a particle around the black hole. As the quantum mechanics in an external field encounters the problem of particle creation leading to many particle systems we feel that a proper approach to the background gravitational field goes through the field quantization. Quantum field theory can be defined by means of Green functions. In the Minkowski space the locality and Poincare invariance determine the Green functions and allow a construction of free quantum fields. In the curved space the Green function is not unique. The non-uniqueness can be interpreted as a non-uniqueness of the physical vacuum [1][2]. There is less ambiguity in the definition of the Green function on the Riemannian manifolds (instead of the physical pseudo-Riemannian ones). The Euclidean approach appeared successful when applied to the construction of

quantum fields on the Minkowski spacetime [3]. We hope that such an approach will be fruitful in application to a curved background as well. In contradistinction to the Minkowski spacetime an analytic continuation of Euclidean fields to quantum fields from the Riemannian metric to the pseudoRiemannian one is possible only if the manifold has an additional reflection symmetry [4][5]. The reflection symmetry may have a physical meaning concerning tunneling phenomena which could justify the requirement of an additional symmetry of the gravitational background [6]. The Euclidean approach to quantum fields on a curved background has been discussed earlier in [7] [8][9] and developed in [4][5].

The event horizon has a crucial relevance for locality of quantum field theory because some information is lost behind the horizon. It is also the defining property of the black hole. The event horizon is a global property of the pseudoRiemannian manifold. Hence, it is hard to see how it could be defined after an analytic continuation to the Riemannian manifold. There is however a proper substitute:the bifurcate Killing horizon [10]. As proved in [11] a manifold with the Killing horizon (a static black hole is an example of the Killing horizon) can be extended to the manifold with a bifurcate Killing horizon. Moreover, there always exists an extension with the wedge reflection symmetry [12] which seems crucial for quantum field theory and for an analytic continuation between pseudoRiemannian and Riemannian manifolds. The bifurcate Killing horizon is a local property which can be treated in local coordinates [10]. In local Kruskal-Szekeres type of coordinates close to the bifurcate Killing horizon the metric tensor $g_{\mu\nu}$ tends to zero at the horizon. This property is preserved after a continuation to the Riemannian metric. We can treat approximately the Riemannian manifold $\mathcal{N}$ with the bifurcate Killing horizon as $\mathcal{N} = \mathcal{R}_2 \times \mathcal{M}_{D-1}$, where $\mathcal{R}_2$ is the two dimensional Rindler space and $\mathcal{M}_{D-1}$ enters the definition of the bifurcate Killing horizon as an intersection of past and future horizons. There are well-known examples of the approximation in the form of a product: the Schwarzschild solution can be approximated near the horizon by a product of the Rindler space and the two-dimensional sphere . However, we do not restrict ourselves to metrics which are solutions of Einstein equations .

We consider an equation for the Green functions in an approximate metric near the bifurcate Killing horizon . We expand the solution into eigenfunctions of the Laplace-Beltrami operator on $\mathcal{M}$. If $\mathcal{M}$ is compact without a boundary then the Laplace -Beltrami operator has a discrete spectrum starting from 0 (the zero mode). We show that the higher modes are damped by a tunneling mechanism. As a consequence the position of the point defining the Green function on the manifold $\mathcal{M}$ becomes irrelevant. The Green function near the bifurcate Killing horizon can be well approximated by the Green function of the two-dimensional conformal free field. The splitting of the Green function near the horizon into a product of the two dimensional function and a function on $\mathcal{M}$ has been predicted by Padmanabhan [13]. However, we obtain its exact form. Moreover, we show that if $\mathcal{M}$ is not compact, but the volume of a geodesic ball of radius r grows like a power of r, then the reduction to a two dimensional

Green function does not take place.

The mechanism of the dimensional reduction presented in this paper may work for other models. In particular, for the ones with a warped form of the metric [14] encountered in the brane models [15]. Such models could explain why the Universe is confined to a submanifold and the probability of tunneling out of it is small.

2 An approximation at the horizon

We consider a $D+1$ dimensional Riemannian manifold $\mathcal{N}$ with a metric g_{AB} characterized by a bifurcate Killing horizon. This notion assumes a symmetry generated by the Killing vector ξ^A. Then, it is assumed that the Killing vector is orthogonal to a (past oriented) D dimensional hypersurface $\mathcal{H}_A$ and a (future oriented) hypersurface $\mathcal{H}_B$ [10]. The Killing vector ξ^A is vanishing (i.e.,$\xi^A(x)\xi_A(x)=0$) on an intersection of $\mathcal{H}_A$ and $\mathcal{H}_B$ defining a $D-1$ dimensional surface $\mathcal{M}$ (which can be described as the level surface $f(x)=const$) . The bifurcate Killing horizon implies that the space-time has locally a structure of the one seen from an accelerated frame, i.e., the structure of the Rindler space. Padmanabhan [13] describes such a bifurcate Killing horizon as a transformation from a local Lorentz frame to the local accelarated (Rindler) frame. In [11] it is proved that the space-time with a Killing horizon can be extended to a space-time with the bifurcate Killing horizon. There, it is also shown that the extension can be chosen in such a way that the "wedge reflection symmetry" [12] is satisfied.In the local Rindler coordinates the reflection symmetry is $(x_0,y,\mathbf{x})\to(x_0,-y,\mathbf{x})$. The symmetry means that the metric splits into a block form

$$ds^2=\sum_{a,b=0,1} g_{ab}dx^a dx^b+\sum_{jk>1} g_{jk}dx^j dx^k$$

The bifurcate Killing horizon distinguishes a two-dimensional subspace of the tangent space. At the bifurcate Killing horizon the two-dimensional metric tensor g_{ab} is degenerate. In the adapted coordinates such that $\xi^A=\partial_0$ we have $g_{10}=0$ and the metric does not depend on x_0. Then, $\det[g_{ab}]\to 0$ at the horizon means that $g_{00}(y=0,\mathbf{x})=0$ or $g_{11}(y=0,\mathbf{x})=0$ (if both of them were zero then the curvature tensor would be singular). We assume $g_{00}(y=0,\mathbf{x})=0$. As g_{00} is non-negative its Taylor expansion must start with y^2. Hence, if we neglect the dependence of the two-dimensional metric g_{ab} on $\mathbf{x}$ then we can write it in the form

$$\begin{aligned} ds_g^2 &\equiv g_{AB}dx^A dx^B=-y^2(dx^0)^2+dy^2+\textstyle\sum_{j,k\geq 2} g_{jk}(y,\mathbf{x})dx^j dx^k \\ &\equiv y^2\Big(-(dx^0)^2+y^{-2}(dy^2+ds^2_{D-1})\Big) \end{aligned} \tag{1}$$

If we neglect the dependence of g_{jk} on y near the horizon then the metric ds^2_{D-1} (denoted ds^2_M) can be considered as a metric on the $D-1$ dimensional surface

$\mathcal{M}$ being the common part of $\mathcal{H}_A$ and $\mathcal{H}_B$. Hence, in eq.(1) $\mathcal{N} = \mathcal{R}_2 \times \mathcal{M}$ where $\mathcal{R}_2$ is the two-dimensional Rindler space. As an example of the approximation of the geometry of $\mathcal{N}$ we could consider the four dimensional Schwarzschild black hole when $\mathcal{N} \simeq \mathcal{R}_2 \times S^2$ (quantum theory with such an approximation is discussed in [16]) .

We shall work with Euclidean version of the metric (1)

$$\begin{aligned} ds^2 &= y^2(dx^0)^2 + dy^2 + \textstyle\sum_{j,k\geq 2} g_{jk}(y,\mathbf{x})dx^j dx^k \\ &\equiv y^2\Big((dx^0)^2 + y^{-2}(dy^2 + ds^2_{D-1})\Big) \end{aligned} \tag{2}$$

It is assumed that this is a Riemannian metric on the Riemannian section of a certain complexified manifold $\mathcal{N}$ [17].

Let

$$\triangle_N = \frac{1}{\sqrt{g}}\partial_A g^{AB}\sqrt{g}\partial_B \tag{3}$$

be the Laplace-Beltrami operator on $\mathcal{N}$.

We are interested in the calculation of the Green functions

$$(-\triangle_N + m_N^2)\mathcal{G}_N^m = \frac{1}{\sqrt{g}}\delta \tag{4}$$

A solution of eq.(4) can be expressed by the fundamental solution of the diffusion equation

$$\partial_\tau P = \frac{1}{2}\triangle_N P$$

with the initial condition $P_0(x,x') = g^{-\frac{1}{2}}\delta(x-x')$. Then

$$\mathcal{G}_N^m = \frac{1}{2}\int_0^\infty d\tau \exp(-\frac{1}{2}m_N^2\tau)P_\tau \tag{5}$$

In order to prove Eq.(5) we multiply the diffusion equation by $\exp(-\frac{1}{2}m_N^2\tau)$ and integrate both sides over τ applying the initial condition for P_τ.

In the approximation (2) we have (if g_{jk} is independent of y)

$$\triangle_N = y^{-2}\partial_0^2 + y^{-1}\partial_y y\partial_y + \triangle_M$$

where $\triangle_M$ is the Laplace-Beltrami operator for the metric

$$ds_M^2 = \sum_{jk} g_{jk}(0,\mathbf{x})dx^j dx^k$$

Then, eq.(4) reads

$$-(\partial_0^2 + y\partial_y y\partial_y + y^2\triangle_M - y^2 m_N^2)\mathcal{G}_N^m = y\frac{1}{\sqrt{g}_M}\delta \tag{6}$$

After an exponential change of coordinates

$$y = b \exp u \tag{7}$$

eq.(6) takes the form

$$\Big(-\partial_0^2 - \partial_u^2 - b^2 \exp(2u)(\triangle_M - m_N^2)\Big)\mathcal{G}_N^m = g_M^{-\frac{1}{2}}\delta(x_0 - x_0')\delta(u - u')\delta(\mathbf{x} - \mathbf{x}') \tag{8}$$

Let us denote

$$\int d\mathbf{x}\sqrt{g_M}\mathcal{G}_N^0 \equiv G(x_0, x_1; x_0', x_1') \tag{9}$$

Then, the integrated Green functions (8)-(9) with $m_N = 0$ are solutions of the equation for the two-dimensional Green function

$$(-\partial_0^2 - \partial_1^2)G = \delta(x_0 - x_0')\delta(x_1 - x_1') \tag{10}$$

3 Generalized hyperbolic manifold $\mathcal{D}$

If $\mathcal{M} = R^n$ $(n = D - 1\ )$ then we have in eq.(2) the Rindler metric [18]

$$ds^2 = y^2(dx^0)^2 + dy^2 + (dx^2)^2 + ... + (dx^D)^2 \tag{11}$$

where $y > 0$ and $\mathbf{x} \in R^{D-1}$. In the Rindler $D+1$ dimensional space the formula (3) for the Laplace-Beltrami operator reads

$$\triangle_R = y^{-2}\partial_0^2 + y^{-1}\partial_y y \partial_y + \partial_2^2 + ... + \partial_D^2 \tag{12}$$

In [19] we have expressed the massless Green function for Rindler space by the massive one for the hyperbolic space. The conformal relation between metrics is responsible for a relation between Green functions. In the same way let us consider a conformal transformation of the metric on $\mathcal{N}$ to the optical metric [20] $\tilde{g}_{AB} \equiv g_{AB}g_{00}^{-1}$ when

$$ds_{op}^2 = \tilde{g}_{AB}dx^A dx^B = (dx^0)^2 + y^{-2}(dy^2 + ds_M^2) \tag{13}$$

The equation for the Green function in the metric (13) is

$$-\Big(\partial_0^2 + y^2\partial_y^2 - (D-2)y\partial_y + y^2\triangle_M - m^2\Big)\mathcal{G}_{op} = y^D g_M^{-\frac{1}{2}}\delta \tag{14}$$

If we introduce $(n = D - 1)$

$$\mathcal{G}_{op}(X, X') = y^{\frac{n}{2}} y'^{\frac{n}{2}} \hat{\mathcal{G}}_{op}(X, X') \tag{15}$$

then

$$-\Big(\partial_0^2 + y\partial_y y\partial_y - \frac{n^2}{4} + y^2\triangle_M - m^2\Big)\hat{\mathcal{G}}_{op} = y g_M^{-\frac{1}{2}}\delta \tag{16}$$

Comparing eqs.(16) and eq.(6) we can see that if $m_N = 0$ and $m^2 = -\frac{n^2}{4}$ then $\mathcal{G}_N$ coincides with $\mathcal{G}_{op}$.

The form of the optical metric (13) suggests that we should study a D dimensional manifold $\mathcal{D} = R \times \mathcal{M}$ with the metric

$$ds_D^2 = y^{-2}(dy^2 + ds_M^2) \tag{17}$$

The equation (4) for the Green function on $\mathcal{D}$ reads

$$-\Big(y^2\partial_y^2 - (D-2)y\partial_y + y^2\triangle_M - m^2\Big)\mathcal{G}_D^m = y^D g_M^{-\frac{1}{2}}\delta \tag{18}$$

or after the transformation (15)

$$-\Big(y\partial_y y\partial_y - \frac{(D-1)^2}{4} + y^2\triangle_M - m^2\Big)\hat{\mathcal{G}}_D = y g_M^{-\frac{1}{2}}\delta \tag{19}$$

We consider the Fourier transform of the Green function (6)

$$\mathcal{G}_N^m(x_0, u, \mathbf{x}; x_0', u', \mathbf{x}') = \int dp_0 \exp(ip_0(x_0 - x_0'))\tilde{\mathcal{G}}_N^m(p_0, u, \mathbf{x}; u', \mathbf{x}') \tag{20}$$

It follows from eqs.(6) and (19) that at $m_N = 0$

$$\tilde{\mathcal{G}}_N^0(p_0, u, u'; \mathbf{x}, \mathbf{x}') = \hat{\mathcal{G}}_D^m(u, u'; \mathbf{x}, \mathbf{x}') \tag{21}$$

if

$$p_0^2 = m^2 + \frac{n^2}{4} \tag{22}$$

We define the operator

$$\mathcal{B} = y\partial_y y\partial_y - \frac{(D-1)^2}{4} + y^2\triangle_M \tag{23}$$

and its heat kernel $\hat{P}$ (this heat kernel is discussed also in [16])

$$\partial_\tau \hat{P}_\tau = \frac{1}{2}\mathcal{B}\hat{P}_\tau \tag{24}$$

with the initial condition $\hat{P}_0(X, X') = y\frac{1}{\sqrt{g_M}}\delta(X - X')$ (here $X = (y, \mathbf{x})$).

If we have the fundamental solution (24) then integrating over τ we can solve the equation for the Green function

$$(-\mathcal{B} + m^2)\mathcal{G}_D^m = y\frac{1}{\sqrt{g}}\delta \tag{25}$$

Applying eqs.(21),(24) and (25) we obtain

$$\tilde{\mathcal{G}}_N^0(p_0, u, u'; \mathbf{x}, \mathbf{x}') = \int_0^\infty d\tau \exp(-\frac{\tau}{2}p_0^2 + \frac{\tau}{8}n^2)\hat{P}_\tau(y, \mathbf{x}; y', \mathbf{x}') \tag{26}$$

or

$$\mathcal{G}_N^0(x_0, u, \mathbf{x}; x_0', u', \mathbf{x}') = \int_0^\infty d\tau (2\pi\tau)^{-\frac{1}{2}} \exp(-\frac{1}{2\tau}(x_0 - x_0')^2 + \frac{\tau}{8}n^2)\hat{P}_\tau(y, \mathbf{x}; y', \mathbf{x}') \tag{27}$$

4 The heat kernel on $\mathcal{D}$

The Laplace-Beltrami operator $\triangle_D$ on $\mathcal{D}$ reads

$$\triangle_D = y^2\partial_y^2 - (D-2)y\partial_y + y^2\triangle_M \tag{28}$$

We consider the heat equation

$$\partial_\tau P_\tau = \frac{1}{2}\triangle_D P_\tau \tag{29}$$

with the initial condition $y^{-D}g_M^{-\frac{1}{2}}\delta$.

If $\mathcal{M}$ is a compact manifold without a boundary then the spectrum of the Laplace-Beltrami operator $\triangle_M$ is discrete [21]

$$-\triangle_M u_k = \epsilon_k u_k \tag{30}$$

The eigenfunctions u_k are normalized

$$\int d\mathbf{x}\sqrt{g}_M \overline{u}_n u_k = \delta_{nk}$$

and satisfy the completeness relation

$$\sum_k \overline{u}_k(\mathbf{x})u_k(\mathbf{x}') = g_M^{-\frac{1}{2}}\delta(\mathbf{x}-\mathbf{x}') \tag{31}$$

Let us note that on a compact manifold a constant is an eigenfunction (30) with the lowest eigenvalue 0 as

$$\triangle_M 1 = 0$$

This zero mode is crucial for the behaviour of the Green function near the bifurcate Killing horizon.

We can expand the fundamental solution of the heat equation (29) in the complete set of eigenfunctions (30)

$$P_\tau(y,\mathbf{x};y',\mathbf{x}') = \sum_k q_\tau^k(y,y')\overline{u}_k(\mathbf{x})u_k(\mathbf{x}') \tag{32}$$

Inserting in eq.(29) we obtain an ordinary differential equation for q_τ^k.

In the Appendix we calculate P_τ assuming that P_τ depends only on the geodesic distance σ on $\mathcal{D}$. First, we show that if σ_M is the geodesic distance on $\mathcal{M}$ then σ is determined by the formula (this is a generalization of the well-known formula for the hyperbolic space)

$$\cosh\sigma = 1 + (2yy')^{-1}(\sigma_M^2 + (y-y')^2) \tag{33}$$

In order to prove eq.(33) we show by means of a direct calculation that σ satisfies the Hamilton-Jacobi equation [22]

$$y^2(\partial_y\sigma\partial_y\sigma + g^{jk}\partial_j\sigma\partial_k\sigma) = 1 \tag{34}$$

if σ_M satisfies the equation

$$g^{jk}\partial_j\sigma_M\partial_k\sigma_M = 1 \tag{35}$$

We return to the heat equation (24) which results by the transformation (15) from the heat equation (29) . We integrate both sides of eq.(24) with respect to the measure $\sqrt{g_M}d\mathbf{x}$ of the manifold $\mathcal{M}$. Denote

$$q_\tau(y, y') = \int \sqrt{g_M}d\mathbf{x}\hat{P}_\tau(y, \mathbf{x}; y', \mathbf{x}') \tag{36}$$

It follows that q_τ satisfies the equation

$$\partial_\tau q_\tau = \frac{1}{2}\Big(\partial_u^2 - \frac{(D-1)^2}{4}\Big)q_\tau \tag{37}$$

with the initial condition $q_0 = \delta(u - u')$ (this is the same equation as for q_τ^0 in eq.(32) after an exponential change of coordinates (7)).

The solution of eq.(37) is

$$q_\tau(u, u') = (2\pi\tau)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2\tau}(u-u')^2 - \frac{\tau}{8}(D-1)^2\Big) \tag{38}$$

When the lhs of eq.(36) is known (from eq.(38)) then eq.(36) can be considered as an integral equation for $\hat{P}_\tau$.

We can check using eqs.(27) and (38) that eq.(10) is satisfied as

$$\begin{aligned}&\int d\mathbf{x}\sqrt{g_M}\mathcal{G}_N^0(y, x_0, \mathbf{x}; y', x_0', \mathbf{x}') = \int_0^\infty d\tau(2\pi\tau)^{-1}\exp(-\tfrac{1}{2\tau}(u-u')^2 - \tfrac{1}{2\tau}(x_0 - x_0')^2)\\ &= -\tfrac{1}{4\pi}\ln\Big((u-u')^2 + (x_0 - x_0')^2\Big)\end{aligned} \tag{39}$$

Let us define the geodesic ball at the point $\mathbf{x}_0$

$$B_r(\mathbf{x}_0) = (\mathbf{y} \in \mathcal{M} : \sigma(\mathbf{x}_0, \mathbf{y}) \le r)$$

We denote

$$\Omega(r) = Vol(B_r(\mathbf{0})) \equiv \int_0^r d\rho\omega(\rho) \tag{40}$$

defining the density $\omega(r)$.

The volume element is

$$d\mathbf{x}\sqrt{g_M}(\mathbf{x}) = dr\omega(r)dS \tag{41}$$

where dS denotes an integral over the surface of the ball (with a normalization $\int dS = 1$). Then, under the assumption that P_τ depends only on the geodesic distance, eq.(36) (rewritten in terms of P of eq.(29)) reads

$$\begin{aligned} &\int_0^\infty dr\omega(r)P_\tau\Big((2yy')^{-1}(r^2+(y-y')^2)\Big) \\ &= (yy')^{\frac{n}{2}}(2\pi\tau)^{-\frac{1}{2}}\exp\Big(-(2\tau)^{-1}(\ln y-\ln y')^2-\tfrac{n^2\tau}{8}\Big) \end{aligned} \tag{42}$$

As an example: heat kernels on homogeneous spaces of rank 1 depend only on the geodesic distance [23][24]. In general, we must treat our assumption as an approximation.

Let us denote

$$\rho = \frac{r^2}{2yy'} \tag{43}$$

Then, eq.(42) can be rewritten as

$$\int_0^\infty P_\tau(\rho+v)f(2yy'\rho)d\rho = T_n(\tau,v) \tag{44}$$

where

$$f(u) = \omega(\sqrt{u})u^{-\frac{1}{2}} \tag{45}$$

and

$$v = (2yy')^{-1}(y-y')^2$$

can be considered as $\cosh\sigma - 1$ at $\mathbf{x} = \mathbf{x}'$. The rhs of eq.(44)

$$T_n(\tau,v) = \frac{1}{2}(yy')^{\frac{n-2}{2}}(2\pi\tau)^{-\frac{1}{2}}\exp\Big(-\frac{1}{2\tau}w^2-\frac{n^2\tau}{8}\Big) \tag{46}$$

is a function of

$$w = \ln\frac{y'}{y}$$

or a function of

$$v = \cosh w - 1 \tag{47}$$

Then, w in eq.(47) is replaced by the geodesic distance, so that in eq.(44) $v = \cosh\sigma - 1$.

When the function ω is known then we obtain an integral equation for $\hat{P}$. On R^n

$$\omega(r) = A(n-1)r^{n-1} \tag{48}$$

where $A(n-1)$ is the area of an $n-1$ dimensional unit sphere.

We can see from eq.(44)-(45) that P_τ is determined by the volume measure Ω. We solve eq.(44) for P_τ in the Appendix under the assumption that the volume $\Omega(r)$ grows like a power of r . For compact manifolds we study the behaviour of P_τ in the next section.

5 Euclidean free fields near the horizon

We investigate in this section the Green function (8) in $D+1$ dimensions under the assumption that $\mathcal{M}$ is $D-1$ dimensional compact manifold without a boundary with a complete set of eigenfunctions (30)-(31). We introduce the complete basis of eigenfunctions in the space $L^2(dx_0dx_1)$ of the remaining two coordinates

$$(-\partial_0^2-\partial_1^2+b^2\epsilon_k\exp(2x_1)\Big)\phi_k^E(x_0,x_1)=E\phi_k^E(x_0,x_1) \tag{49}$$

In eq.(49) E denotes the set of all the parameters the solution ϕ^E depends on. The solutions ϕ satisfy the completeness relation

$$\int d\nu(E)\overline{\phi}_k^E(x_0,x_1)\phi_k^E(x_0',x_1')=\delta(x_0-x_0')\delta(x_1-x_1') \tag{50}$$

with a certain measure ν and the orthogonality relation

$$\int dx_0dx_1\overline{\phi}_k^E(x_0,x_1)\phi_k^{E'}(x_0,x_1)=\delta(E-E') \tag{51}$$

where again the δ function concerns all parameters characterizing the solution. Then, we expand the Green function into the Kaluza-Klein modes u_k as in eq.(32)

$$\begin{aligned}&\mathcal{G}_N(x_0,x_1,\mathbf{x};x_0',x_1',\mathbf{x}')\\&=\textstyle\sum_k g_k(x_0,x_1;x_0',x_1')\overline{u}_k(\mathbf{x})u_k(\mathbf{x}')\end{aligned} \tag{52}$$

where

$$g_k(x_0,x_1;x_0',x_1')=\int d\nu(E)E^{-1}\overline{\phi}_k^E(x_0,x_1)\phi_k^E(x_0',x_1') \tag{53}$$

g_k is the kernel of the inverse of the operator

$$H_k=-\partial_0^2-\partial_1^2+b^2\epsilon_k\exp(2x_1)$$

i.e.,

$$H_kg_k(x_0,x_1;x_0',x_1')=\delta(x_0-x_0')\delta(x_1-x_1')$$

The integral over $\mathbf{x}$ eliminates all u_k from the sum (52) except of $k=0$. Hence,

$$\begin{aligned}&G(x_0,x_1;x_0',x_1')=\textstyle\int d\nu(E)E^{-1}\overline{\phi}_0^E(x_0',x_1')\phi_0^E(x_0,x_1)\\&=-\tfrac{1}{4\pi}\ln\Big((x_0-x_0')^2+(x_1-x_1')^2\Big)\end{aligned} \tag{54}$$

In the limit $x_1\to+\infty$ the eigenfunctions ϕ^E decay exponentially (except of the zero mode) when entering the potential barrier. Hence, we may expect that the Green functions also decay exponentially from the zero mode contribution $\mathcal{G}_0$ away from the horizon .

We study this phenomenon in more detail now. First, we write the solution of eq.(49) in the form

$$\phi_k^E = \exp(ip_0x_0)\phi_k^{p_1}(x_1) \tag{55}$$

where

$$(-\partial_1^2 + b^2\epsilon_k \exp(2x_1))\phi_k^{p_1} = p_1^2\phi_k^{p_1} \tag{56}$$

Now, $E = p_0^2 + p_1^2$ and $d\nu = dp_0dp_1$ in eqs.(49)-(51). When $\epsilon_k = 0$ then the solution of eq.(56) is the plane wave

$$\phi_0^{p_1} = (2\pi)^{-\frac{1}{2}} \exp(ip_1x_1)$$

Hence, eq.(54) follows by an explicit calculation.

The normalized solution of eq.(56) which behaves like a plane wave with momentum p_1 for $x_1 \to -\infty$ and decays exponentially for $x_1 \to +\infty$ reads

$$\phi_k^{p_1} = N_{p_1}K_{ip_1}(b\sqrt{\epsilon_k}\exp(x_1)) \tag{57}$$

where K_ν is the modified Bessel function of the third kind of order ν [25].

This solution is inserted into the formula (52) for the Green function with the normalization (51)

$$\int_{-\infty}^{\infty} dx_1\overline{\phi}_k^{p_1}(x_1)\phi_k^{p_1'}(x_1) = \delta(p_1 - p_1') \tag{58}$$

Hence (see [26]),

$$N_{p_1}^2 = p_1\sinh(\pi p_1)\frac{2}{\pi^2} \tag{59}$$

Then, performing the integral over p_0 in eq.(53)

$$\int dp_0 \exp(ip_0(x_0 - x_0'))(p_0^2 + p_1^2)^{-1} = \pi|p_1|^{-1}\exp(-|p_1||x_0 - x_0'|)$$

we obtain (G is defined in eq.(54))

$$\begin{aligned}&\mathcal{G}_N(x_0, x_1, \mathbf{x}; x_0', x_1', \mathbf{x}') - G(x_0, x_1; x_0', x_1') = \tfrac{4}{\pi^2}\textstyle\int_0^\infty dp_1 \sinh(\pi p_1)\exp(-p_1|x_0 - x_0'|)\\ &\textstyle\sum_{k\neq 0} K_{ip_1}(b\sqrt{\epsilon_k}\exp(x_1))K_{ip_1}(b\sqrt{\epsilon_k}\exp(x_1'))\overline{u}_k(\mathbf{x})u_k(\mathbf{x}')\end{aligned} \tag{60}$$

An estimate of the sum over k is difficult in general. The sum itself should be understood in the sense of a convergence of bilinear forms (or equivalently in the sense of a convergence of the partial sum as a distribution). There are some subtleties in this convergence. For example, if we let $b \to 0$ then the solution of eq.(8) ($u = x_1, m_N = 0$) is

$$\mathcal{G}_N(x_0, x_1, \mathbf{x}; x_0', x_1', \mathbf{x}') = G(x_0, x_1; x_0', x_1')g_M^{-\frac{1}{2}}\delta(\mathbf{x} - \mathbf{x}')$$

The limit $x_1 \to -\infty$ could possibly be realized as $b \to 0$. In the latter limit the rhs of eq.(60) should tend to

$$G(x_0, x_1; x_0', x_1') g_M^{-\frac{1}{2}} \sum_{k \neq 0} \overline{u}_k(\mathbf{x}) u_k(\mathbf{x}')$$

It is difficult to prove such limits in general.

We shall restrict ourselves to particular examples and heuristic arguments for some estimates of the rhs of eq.(60) which however cannot be uniform in $x_1 \to -\infty$. Let us consider the simplest example $\mathcal{M} = S^1$. Then, $u_k(x_2) = (2\pi)^{-\frac{1}{2}} \exp(ikx_2)$ and $\epsilon_k = k^2$. The sum over k can be performed by means of the representation of the Bessel function

$$K_{i\nu}(z) = \int_0^\infty dt \exp(-z \cosh t) \cos(\nu t) \tag{61}$$

We have

$$\begin{gathered}
\sum_{k=1}^\infty \exp\Big(- kz\cosh(t) - kz'\cosh(t')\Big) \cos(k(x_2 - x_2')) = \\
\Re\Big(\exp\Big(- z\cosh(t) - z'\cosh(t') + i(x_2 - x_2')\Big) \\
\Big(1 - \exp(-z\cosh(t) - z'\cosh(t') + i(x_2 - x_2'))\Big)^{-1}\Big)
\end{gathered} \tag{62}$$

Inserting in eq.(60) ($z = by$) and approximating the denominator in eq.(62) by 1 we obtain an asymptotic estimate for large y and y' . Then,

$$\begin{gathered}
\mathcal{G}_N(x_0, x_1, \mathbf{x}; x_0', x_1', \mathbf{x}') - G(x_0, x_1; x_0', x_1') \\
\simeq \frac{8}{\pi^2} \cos(x_2 - x_2') \int_0^\infty dp_1 \sinh(\pi p_1) \exp(-p_1|x_0 - x_0'|) K_{ip_1}(b\exp(x_1)) K_{ip_1}(b\exp(x_1'))
\end{gathered} \tag{63}$$

We can estimate the integral over p_1 if $|x_0 - x_0'| > \pi$. In such a case inserting the asymptotic expansion of the Bessel function we obtain

$$\begin{gathered}
\mathcal{G}_N(x_0, x_1, \mathbf{x}; x_0', x_1', \mathbf{x}') - G(x_0, x_1; x_0', x_1') \\
\simeq \frac{8}{\pi^2} \cos(x_2 - x_2')(|x_0 - x_0'| - \pi)^{-1} \exp\Big(- b(\exp(x_1) + \exp(x_1'))\Big)
\end{gathered} \tag{64}$$

It follows that $\mathcal{G}_N$ tends exponentially fast to the Green function for the two-dimensional quantum field theory. $\mathcal{G}_N$ does not vanish at the horizon. We could impose Dirichlet boundary conditions on $\mathcal{G}_N$ at the horizon. For example we could demand that $\mathcal{G}_N$ is vanishing on the line $x_1 = u$ (as suggested in [27]). We can construct such a Green function by means of the method of images. When $u \to -\infty$ and $x_1 \to -\infty$ (still $x_1 > u$) then the Dirichlet Green function $\mathcal{G}_N^D$ tends to the two-dimensional Dirichlet Green function of the two-dimensional Laplace operator vanishing on the line $x_1 = u$. However, as pointed in [27] it is not obvious that we should impose such a boundary condition at the horizon.

We can derive an integral representation for the Green function for the n-dimensional torus $S^1 \times S^1 \times .... \times S^1$ (then $\epsilon_k = k_1^2 + ... + k_n^2$) applying the formula

$$\begin{aligned}&\exp(-by\cosh(t)\sqrt{k_1^2+...+k_n^2})\\&=\pi^{-\frac{1}{2}}\int_0^\infty drr^{-2}\exp(-\tfrac{1}{4r^2})\exp\Big(-r^2y^2b^2\cosh^2(t)(k_1^2+...+k_n^2)\Big)\end{aligned} \tag{65}$$

Now the sum over $k_1, ..., k_n$ can be performed. It is expressed by a product of elliptic θ-functions. A precise analysis of such a formula is still difficult. We shall rely on an approximation applicable to general compact $\mathcal{M}$ (for the torus the Weyl approximation of the spectrum, applied in eq.(67) below is exact).

We estimate the rhs of eq.(60) for large x_1 and x_1' by means of a simplified argument applicable when $\mathbf{x} = \mathbf{x}'$, $x_1 = x_1'$, $|u(\mathbf{x})| \leq C$. Then,

$$\begin{aligned}&|\mathcal{G}_N(x_0,x_1,\mathbf{x};x_0',x_1,\mathbf{x})-G(x_0,x_1;x_0',x_1)|\\&\leq C^2\tfrac{4}{\pi^2}\int_0^\infty dp_1\sinh(\pi p_1)\exp(-p_1|x_0-x_0'|)\sum_{k\neq 0}|K_{ip_1}(b\sqrt{\epsilon_k}\exp(x_1))|^2\end{aligned} \tag{66}$$

For the behaviour of the rhs of eq.(66) at large x_1 only large k in the sum (66) are relevant (the finite sum over k is decaying exponentially). For large eigenvalues ($\epsilon_k \geq n$ with n sufficiently large) we can apply the Weyl approximation for the eigenvalues distribution [28] with the conclusion

$$\begin{aligned}&|\mathcal{G}_N(x_0,x_1,\mathbf{x};x_0',x_1,\mathbf{x})-G(x_0,x_1;x_0',x_1)|\\&\leq C^2\tfrac{4}{\pi^2}\int_0^\infty dp_1\sinh(\pi p_1)\exp(-p_1|x_0-x_0'|)\sum_{\delta\leq\epsilon_k\leq n}|K_{ip_1}(b\sqrt{\epsilon_k}\exp(x_1))|^2\\&+R\int_0^\infty dp_1\sinh(\pi p_1)\exp(-p_1|x_0-x_0'|)\int_{|\mathbf{k}|\geq\sqrt{n}}d\mathbf{k}|K_{ip_1}(b|\mathbf{k}|\exp(x_1))|^2\end{aligned} \tag{67}$$

where δ is the lowest non-zero eigenvalue.

The finite sum as well as the integral on the rhs of eq.(67) are decaying exponentially. This follows from estimates on Bessel functions $K_\nu(z)$ for large values of z [25]. The analogous formula for $\mathcal{M} = R^n$ (see eq.(71) at the end of this section and ref. [19]) does not lead to the logarithmic term G on the lhs of eq.(60) because $\delta = 0$ in this case (there is no gap in the spectrum and no zero mode).

We introduce now a free Euclidean field as a random field with the correlation function equal to the Green function $\mathcal{G}_N$ (see [5]; we need $\int f\mathcal{G}_N f \geq 0$)

$$\begin{aligned}\Phi(x_0,x_1,\mathbf{x})&=\int dp_0dp_1\sum_k a_k(p_0,p_1)\phi_k^{p_1}(x_1)u_k(\mathbf{x})\exp(ip_0x_0)\\&=\Phi_0(x_0,x_1)+\sum_{k>0}\Phi_k(x_0,x_1,\mathbf{x})\end{aligned} \tag{68}$$

where

$$\langle\overline{a}_k(p_0,p_1)a_{k'}(p_0',p_1')\rangle=\delta_{kk'}\delta(p_0-p_0')\delta(p_1-p_1')(p_0^2+p_1^2)^{-1} \tag{69}$$

Φ_0 is two-dimensional conformal invariant free field with the correlation function

$$\langle\Phi_0(x_0,x_1)\Phi_0(x_0',x_1')\rangle=G(x_0,x_1;x_0',x_1')=-\frac{1}{4\pi}\ln\Big((x_0-x_0')^2+(x_1-x_1')^2\Big) \tag{70}$$

The correlation functions of Φ_k are decaying exponentially. This is so, because for large x_1 the eigenfunctions $\phi_k^{p_1}$ are solutions behind the potential barrier. Hence, the field $\Phi(x_0,x_1,\mathbf{x})$ will decrease exponentially fast for $x_1 \rightarrow +\infty$.

Finally, let us still write down the formulas for $\mathcal{M} = R^{D-1}$ (the conventional Rindler space discussed in [1][29][30] and our earlier paper [19]). Then, the spectrum of the Laplace-Beltrami operator on $\mathcal{M}$ is continuous and instead of the sum over k we have an integral over the momenta $\mathbf{p}$

$$\begin{aligned}&\mathcal{G}_N(x_0,x_1,\mathbf{x};x_0',x_1',\mathbf{x}')\\&=(2\pi)^{-D+1}\int dp_1 d\mathbf{p}|p_1|^{-1}\exp(-|p_1||x_0-x_0'|)\overline{\phi}^{p_1}_{\mathbf{p}}(x_1')\phi^{p_1}_{\mathbf{p}}(x_1)\exp(i\mathbf{p}(\mathbf{x}-\mathbf{x}'))\end{aligned} \tag{71}$$

$\phi^{p_1}_{\mathbf{p}}(x_1)$ is defined in eq.(57) where ϵ_k is replaced by $\mathbf{p}^2$. Then, in order to obtain the formula for the Green function (71) $u_k(\mathbf{x})$ in eq.(52) are replaced by the plane waves $(2\pi)^{-\frac{D}{2}+\frac{1}{2}}\exp(i\mathbf{p}\mathbf{x})$.

The expansion of the Euclidean field takes the form

$$\Phi(x_0,x_1,\mathbf{x}) = (2\pi)^{-\frac{D}{2}+\frac{1}{2}}\int dp_1 d\mathbf{p} a(p_0,p_1,\mathbf{p})\phi^{p_1}_{\mathbf{p}}(x_1)\exp(i\mathbf{p}\mathbf{x}+ip_0x_0) \tag{72}$$

and

$$\langle\overline{a}(p_0,p_1,\mathbf{p})a(p_0',p_1',\mathbf{p}')\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta(p_0-p_0')\delta(p_1-p_1')(p_0^2+p_1^2)^{-1}$$

We can continue analytically the Green functions to the real time. The Green functions define quantum fields if they satisfy some positivity conditions (reflection positivity on the Riemannian manifold [4][5]and Wightman positivity on the pseudoRiemannian manifold). The Euclidean and quantum fields satisfy the identity (true for Riemannian as well as pseudoRiemanian metrics)

$$\Phi(x) = -\int dx'\sqrt{g}\Big(\triangle_N(x')\mathcal{G}_N(x,x')\Big)\Phi(x') \tag{73}$$

Integrating by parts in eq.(73) we express $\Phi(x)$ by the value of $\Phi(x)$ on the boundary (at infinity) where according to eqs.(60) and (68) only the two-dimensional free field Φ_0 on R^2 contributes (because Φ_k vanish at the boundary for $k>0$) and the field $\triangle_N\Phi$ (which is zero if Φ is the free massless field continued analytically to the pseudoRiemannian manifold) . The expression of Φ by its zero mode Φ_0 indicates that we could construct the quantum field in the Fock space of the massless two-dimensional quantum free field. We can obtain a representation of the Virasoro algebra in this Hilbert space.

The formal analytic continuation of free Euclidean fields $x_0 \rightarrow ix_0$ (a generalization of the quantum fields on the Rindler space of refs. [1][29][30]) reads

$$\begin{aligned}\Phi(x_0,x_1,\mathbf{x}) &= \int dp_1\sum_k a_k(p_1)\phi_k^{p_1}(x_1)u_k(\mathbf{x})\exp(-i|p_1|x_0)\\&+\int dp_1\sum_k a_k^+(p_1)\phi_k^{p_1}(x_1)\overline{u_k}(\mathbf{x})\exp(i|p_1|x_0)\end{aligned} \tag{74}$$

where a and a^+ are annihilation and creation operators satisfying the commutation relations

$$[a_k(p_1),a_l^+(p_1')] = \delta_{kl}\delta(p_1-p_1') \tag{75}$$

6 Conclusions

We have shown in sec.5 that if the manifold $\mathcal{M}$ defining the bifurcate Killing horizon is compact then near the horizon of $\mathcal{N}$ the massless Green function of $\mathcal{N}$ can be approximated by the Green function of the two-dimensional massless free field. If the manifold $\mathcal{M}$ is not compact, but rather resembles the flat space in its volume growth (see eq.(A.1)), then we cannot expect the reduction to a two-dimensional model. We cannot apply the eigenfunction expansion of sec.5 but we refer to the results of the Appendix. Let us consider the behaviour at the horizon when $y' = y = \exp(x_1) \to 0$, i.e., $x_1 \to -\infty$. In this limit, if simultaneously $\sigma_M(\mathbf{x}, \mathbf{x}') << yy'$ (without this assumption the limiting behaviour of Green functions would not depend on $\mathbf{x}$) then for $D = 3$ ($k = 0$ in eq.(A.22)) for any $x_0 - x_0'$

$$\mathcal{G} = y^{-1}y'^{-1}\sigma(\sinh\sigma)^{-1}\Big(\sigma^2 + (x_0 - x_0')^2\Big)^{-1} \to \sigma_M^{-2} \tag{76}$$

We can show that eqs.(A.22)-(A.23) imply a generalization of eq.(76) to $D+1$ dimensional manifold $\mathcal{N}$, where for $x_1, x_1' \to -\infty$

$$\mathcal{G}_N(x_0 - x_0', \sigma) \simeq \sigma_M^{-D+1} \tag{77}$$

It is interesting that this limiting behaviour does not depend on x_0 .

Let us consider the case $\mathbf{x} \simeq \mathbf{x}'$ in more detail . Then, $\sigma \simeq |x_1 - x_1'|$ and from eqs.(A.22)-(A.23) for small x_1 and x_1'

$$\mathcal{G}^0_N(x_0 - x_0', \sigma) \simeq \Big((x_1 - x_1')^2 + (x_0 - x_0')^2\Big)^{-\frac{D}{2}+\frac{1}{2}} \tag{78}$$

For arbitrary x_1 and x_1' we have directly from eq.(A.22)

$$\mathcal{G}^0_N(x_0 - x_0', \sigma)_{2k+2} = \exp\left(-(k+1)(x_1' + x_1)\right)\Big(\sinh(x_1 - x_1')^{-1}\tfrac{d}{dx_1}\Big)^k \\ (x_1 - x_1')(\sinh(x_1 - x_1'))^{-1}\Big((x_1 - x_1')^2 + (x_0 - x_0')^2\Big)^{-1} \tag{79}$$

It is decaying exponentially for $x_1 \to +\infty$ and $x_1' \to +\infty$. This behaviour results from the tunnelling through the barrier described by the quantum mechanical equation (56). We can obtain from eq.(A.23) the exponential decay in x_1 also for the odd case ($n = 2k+1$) but a derivation needs some detailed estimates.

The behaviour (77) is different than the one of eq.(67) (which is suggested in [13]). We can conclude that if the Laplace-Beltrami operator on the manifold $\mathcal{M}$ has a discrete spectrum then the Green function on the manifold $\mathcal{N}$ close to the bifurcate Killing horizon is approximately logarithmic like in the two-dimensional massless free field theory. If however the spectrum of the Laplace-Beltrami operator on $\mathcal{M}$ is continuous then such an approximation is invalid. In this case we obtain a power like behaviour at the horizon (eq.(77)) as for the conventional D+1 massless Euclidean field theory in a flat space. Away from the

horizon we have an exponential decay of the Green function $\mathcal{G}_N$ characteristic to the tunneling through the barrier. In the case of the discrete spectrum of $\triangle_M$ the zero mode of the compact manifold $\mathcal{M}$ is not damped by the barrier. As a result its contribution to the Green function $\mathcal{G}_N$ is separated from the tunneling modes leading to the two-dimensional behaviour of the Green function at the bifurcate Killing horizon. The two-dimensional behaviour indicates the relevance of the infinite dimensional conformal group for manifolds with a bifurcate Killing horizon (as briefly discussed at the end of sec.5; it has been shown first in another way in [31]). A realization of the conformal group in the Fock space of quantum fields on the horizon is discussed in [32]. We intend to continue a study of the conformal invariance in the way suggested at the end of sec.5 in a forthcoming publication.

7 Appendix:The Green function of $\mathcal{D}$ with a power-like increasing volume of $\mathcal{M}$

It is difficult to determine the volume growth $\Omega(r)$ (40) in general. We restrict ourselves to a class of manifolds [33](with non-negative Ricci curvature) such that

$$aA(n-1)r^{n-1} \leq \omega(r) \leq A(n-1)r^{n-1} \tag{A.1}$$

with $0 < a \leq 1$. We show in this Appendix that the volume growth determines lower and upper bounds on the Green functions. The index n in eq.(A.1) does not need to be related to the dimension of the manifold. We show that the index determines the behaviour of the heat kernel and the Green functions (for other derivations of some estimates on Green functions from the volume growth see [34]). Applying the inequality (A.1) we can conclude from eq.(44) (as P_τ and f in eq.(44) are positive) that

$$\begin{aligned} &aA(n-1)\int d\rho P_\tau(\rho+v)(\sqrt{2yy'\rho})^{n-2} \leq \int_0^\infty P_\tau(\rho+v)f(2yy'\rho)d\rho \\ &= T_n(\tau,v) \leq A(n-1)\int_0^\infty d\rho P_\tau(\rho+v)(\sqrt{2yy'\rho})^{n-2}d\rho \end{aligned} \tag{A.2}$$

We solve the inequalities (A.2) with respect to the heat kernel. We denote the solution of the inequality by p_τ. Then,

$$ap_\tau \leq P_\tau \leq p_\tau \tag{A.3}$$

and for Green functions

$$a\mathcal{G}_H^m \leq \mathcal{G}_D^m \leq \mathcal{G}_H^m \tag{A.4}$$

When $a = 1$ then $\mathcal{M} = R^n$ and the manifold $\mathcal{D}$ is the hyperbolic space . In such a case the heat kernel p_τ and the Green function $\mathcal{G}_H$ are the ones of the hyperbolic space. Hence, in eqs.(A.3)-(A.4) we estimate the heat kernels and Green functions on $\mathcal{D}$(as functions of the invariant distance) by the heat kernels and Green functions for the hyperbolic space.

Let p_τ be the solution of eq.(44) with f defined in eq.(45) resulting from the upper bound (A.2)(the one for the hyperbolic space) . Then, for an even $n = 2k$ the equation for p_τ reads

$$\begin{array}{l} 2^{k-1}A(n-1)\int_0^\infty p_\tau(\rho+v)\rho^{k-1}d\rho = \\ (2\pi\tau)^{-\frac{1}{2}}\exp\Big(-\frac{1}{2\tau}w^2-\frac{n^2\tau}{8}\Big) \equiv F_n(v) \end{array} \tag{A.5}$$

We assume that $p_\tau(x+v)$ is a k-th derivative of a certain function F_n. Then, integrating by parts in eq.(A.5) we find that this function is F_n. More precisely

$$p_\tau(v) = (-2\pi)^{-k}\frac{d^k}{dv^k}F_{2k}(\tau,v) \tag{A.6}$$

The formula for an odd $n = 2k+1$ is not so simple. The equation for p_τ reads

$$2^{k-\frac{1}{2}}A(2k)\int d\rho p_\tau(\rho+v)\rho^{k-\frac{1}{2}} = F_{2k+1}(v) \tag{A.7}$$

First, let us consider $k = 0$. We set $\rho = \frac{1}{2}\alpha^2$.Then, eq.(A.7) takes the form

$$\int_0^\infty d\alpha p_\tau(\frac{1}{2}\alpha^2+v) = F_1(\tau,v)$$

Differentiating over v

$$\int d\alpha p'_\tau(\frac{1}{2}\alpha^2+v) = F'_1(\tau,v)$$

Shifting v by $\frac{1}{2}\gamma^2$ and integrating over γ we obtain

$$\int d\alpha d\gamma p'_\tau(\frac{1}{2}(\alpha^2+\gamma^2)+v) = \int d\gamma F'_1(\tau,v+\frac{1}{2}\gamma^2) \tag{A.8}$$

In order to perform the integral on the lhs we introduce the cylindrical coordinates $\alpha = r\cos\phi$ and $\gamma = r\sin\phi$. Then, the integral (A.8) is equal to

$$p_\tau(v) = -\frac{1}{\pi}\int_{-\infty}^\infty d\gamma F'_1(\tau,v+\frac{1}{2}\gamma^2) \tag{A.9}$$

For $n = 2k+1$ we have (here $p_\tau^{(0)} = p_\tau$)

$$A(2k)\int_0^\infty p_\tau^{(k)}(\frac{1}{2}\alpha^2+v)\alpha^{2k}d\alpha = F_{2k+1}(\tau,v) \tag{A.10}$$

Assume that

$$p_\tau^{(k)}(v) = \frac{d^k}{dv^k}h_k(\tau,v) \tag{A.11}$$

Then, integrating by parts we obtain from eq.(A.10)

$$A(2k)(-1)^{k+1}(2k-1)\times(2k-3)\times...\times 3\int_0^\infty d\alpha h_k(\tau,\frac{1}{2}\alpha^2+v)=F_{2k+1}(\tau,v) \tag{A.12}$$

Hence, we reduced the problem to the one for $k=0$. Eqs.(A.9) and (A.12) lead to the result

$$p_\tau^{(k)}(v)=(-2\pi)^{-k+1}\int_{-\infty}^{\infty}d\gamma\frac{d^{k+1}}{dv^{k+1}}F_{2k+1}(\tau,\frac{1}{2}\gamma^2+v) \tag{A.13}$$

We can express $p_\tau^{(0)}$ (eq.(A.9)) in the McKean form [35] changing the integration variable γ into r where $v+\frac{\gamma^2}{2}=\cosh r-1$. Then,

$$p_\tau^{(0)}(\sigma)=\exp(-\frac{\tau}{8})\sqrt{2}(2\pi\tau)^{-\frac{3}{2}}\int_\sigma^\infty(\cosh r-\cosh\sigma)^{-\frac{1}{2}}r\exp(-\frac{r^2}{2\tau})dr \tag{A.14}$$

The solutions (A.6) and (A.13) are equal to the ones for the heat kernel on the hyperbolic space. According to eq.(A.4) as functions of the invariant distance (33) on $\mathcal{D}$ these expressions approximate the heat kernel on $\mathcal{D}$. Let us write down these formulas in a more explicit form. We have for odd dimensions $D=n+1=2k+3$ ($k=0,1,..$)

$$p_\tau^{(k+1)}(\sigma)=(-2\pi)^{-k}\exp(-\frac{n^2}{8}\tau+\frac{1}{2}\tau)\Big((\sinh\sigma)^{-1}\frac{d}{d\sigma}\Big)^k p_\tau^{(1)}(\sigma) \tag{A.15}$$

with

$$p_\tau^{(1)}(\sigma)=(2\pi\tau)^{-\frac{3}{2}}\sigma(\sinh\sigma)^{-1}\exp(-\frac{\tau}{2}-\frac{\sigma^2}{2\tau}) \tag{A.16}$$

In even dimensions $D=n+1=2k+2$

$$p_\tau^{(k)}(\sigma)=2\exp(-\tfrac{n^2\tau}{8}+\tfrac{\tau}{8})(-2\pi)^{-k}\Big((\sinh\sigma)^{-1}\tfrac{d}{d\sigma}\Big)^k p_\tau^{(0)}(\sigma) \tag{A.17}$$

These formulas have been derived in another way in [36][37][38][23].

Then, the massless Green function on the hyperbolic space is ($k=0,1,...$)

$$\mathcal{G}_H^m(y,\mathbf{x};y',\mathbf{x}')_{2k+2}=(-2\pi)^{-k}\Big((\sinh\sigma)^{-1}\frac{d}{d\sigma}\Big)^k\sigma(\sinh\sigma)^{-1}\exp(-\nu\sigma) \tag{A.18}$$

where

$$\nu=\sqrt{\frac{n^2}{4}+m^2} \tag{A.19}$$

We obtain analogous formulas for the Green functions for an odd n

$$\begin{aligned}&\mathcal{G}_D^m(y,\mathbf{x};y',\mathbf{x}')_{2k+1}\\&=2\sqrt{2}(2\pi)^{-\frac{3}{2}}(-2\pi)^{-k}\Big((\sinh\sigma)^{-1}\tfrac{d}{d\sigma}\Big)^k\int_\sigma^\infty(\cosh r-\cosh\sigma)^{-\frac{1}{2}}\exp(-\nu r)dr\\&=2(2\pi)^{-\frac{3}{2}}(-2\pi)^{-k}\Big((\sinh\sigma)^{-1}\tfrac{d}{d\sigma}\Big)^k Q_{\nu-\frac{1}{2}}(\cosh\sigma)\end{aligned} \tag{A.20}$$

where ([25])

$$Q_\mu(\cosh\sigma) = \int_\sigma^\infty (2\cosh r - 2\cosh\sigma)^{-\frac{1}{2}} \exp(-\frac{(2\mu+1)r}{2})dr$$

The Fourier transform in x_0 of the massless Green function (6) on $\mathcal{N}$ is equal to the Green function on $\mathcal{D}$ with (see eq.(21))

$$\nu = |p_0|$$

in eq.(A.18),i.e., for an odd dimension

$$\tilde{\mathcal{G}}^0_N(p_0, y, \mathbf{x}; y', \mathbf{x}')_{2k+2} = (-2\pi)^{-k} y^{-k-1} y'^{-k-1} \Big((\sinh\sigma)^{-1}\frac{d}{d\sigma}\Big)^k \sigma(\sinh\sigma)^{-1}\exp(-|p_0|\sigma) \tag{A.21}$$

The massless Green function in D+1 dimensional manifold $\mathcal{N}$, when $D = n+1 = 2(k+1)+1$, can be obtained either from eq.(A.21) by means of the Fourier transform or from eq.(27) by a calculation of the τ-integral

$$\mathcal{G}^0_N(x_0 - x'_0, \sigma)_{2k+2} = y^{-k-1} y'^{-k-1} \Big((\sinh\sigma)^{-1}\frac{d}{d\sigma}\Big)^k \sigma(\sinh\sigma)^{-1}\Big(\sigma^2 + (x_0 - x'_0)^2\Big)^{-1} \tag{A.22}$$

In even dimensions $D = 2k+2$ the formula is more complicated . From eq.(A.20) we obtain

$$\begin{array}{l} \mathcal{G}^0_N(x_0 - x'_0, \sigma)_{2k+1} = y^{-\frac{k+1}{2}} y'^{-\frac{k+1}{2}} \Big((\sinh\sigma)^{-1}\frac{d}{d\sigma}\Big)^k \\ \int_\sigma^\infty (\cosh r - \cosh\sigma)^{-\frac{1}{2}} r(r^2 + (x_0 - x'_0)^2)^{-1} dr \end{array} \tag{A.23}$$

In order to obtain approximate formulas for the generalized hyperbolic manifold $\mathcal{D}$ and the Rindler-type manifold $\mathcal{N}$ with the horizon we should insert in eqs.(A.14)-(A.23) the expression for the invariant distance σ (33) on $\mathcal{D}$.

References

[1] S.A. Fulling, Phys.Rev.**D7**,2850(1973),
S.A. Fulling, J.Phys.**A10**,917(1977)

[2] N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space, Cambridge University Press,1982

[3] A. Jaffe and J. Glimm, Quantum Physics, Springer,1981

[4] J. Dimock, Rev.Math.Phys.**16**,243(2004)

[5] A.Jaffe and G. Ritter, Commun.Math.Phys,**270**,545(2007)

[6] G. W. Gibbons and J.B. Hartle, Phys.Rev.**D42**,2458(1990)
G.W. Gibbons and H.-J. Pohle, Nucl.Phys.**B410**,117(1993)

[7] R.M. Wald, Commun.Math.Phys.**70**,221(1979)

[8] G.F.de Angelis, D.de Falco and G.Di Genova,
Commun.Math.Phys.**103**,297(1986)

[9] Z. Haba, "Reflection positivity and quantum fields on a Riemann surface" in Stochasti Processes, Physics and Geometry, ed. S.Albeverio et al, World Scientific,1990

[10] R.M. Wald, Quantum Field Theory In Curved Spacetime and Black Hole Thermodynamics, The University of Chicago Press,1994

[11] I. Racz and R.M. Wald, Class.Quant.Grav.**9**,2643(1992)

[12] B.S. Kay and R.M. Wald,Phys.Rep.**207**,49(1991)

[13] T. Padmanabhan, Mod.Phys.Lett.**A17**,923(2002), arXiv:gr-qc/0202078

[14] J. Choi, J.Math.Phys.**41**,8163(2000);
J. Choi and S.-T.Hong,J.Math.Phys.**45**,642(2004)

[15] G.Dvali,G.Gabadadze and M.Poratti, Phys.Lett.**485B**,208(2000)
L.Randall, R.Sundrum, Phys.Rev.Lett. **83**3370(1999);
Phys.Rev.Lett.**83**,4690(1999)

[16] A.A. Bytsenko, G. Cognola and S. Zerbini, Nucl.Phys.**458**,267(1996)

[17] P.T.Chrusciel, Acta Phys.Polon.**36**,17(2005)

[18] W. Rindler, Am. J.Phys.**34**,1174(1966)

[19] Z.Haba,J.Phys.**A40**,8535(2007)
E. Heintze and H. Karcher, Ann.scient.Ec.Norm.Sup.**11**,451(1978)

[20] G.W. Gibbons and M.J. Perry, Proc.Roy.Soc.London,**A358**467(1978)

[21] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, New York,1984

[22] B.S.DeWitt, Phys. Rev.**162**,1239(1967);Phys.Rep.**19**,295)(1975)

[23] R. Camporesi, Phys.Rep.**196**,1(1990)

[24] A. Debiard and B. Gaveau, Can.J.Math.**39**,1281(1987)

[25] I.S. Gradshtein and I.M. Ryzhik, Table of Integrals, Series and Products, Academic Press,New York,1962

[26] A.deSessa, Journ.Math.Phys.**15**,1892(1974)
C.M. Sommerfield,Ann.Phys.**84**,285(1974)

[27] L. Susskind and J. Lindsay, An Introduction to Black Holes, Information and the String Theory Revolution, World Scientific, Singapore,2005

[28] M.E. Taylor,Pseudodifferential 0perators, Princeton University Press,1981

[29] D.G. Boulware,Phys.Rev.**D11**,1404(1975)

[30] D.W. Sciama, P. Candelas and D. Deutsch,Adv.Phys.**30**,327(1981)

[31] S. Carlip, Phys.Rev.Lett.**82**,2828(1999)

[32] V. Moretti and N. Pinamonti, Journ.Math.Phys.**45**,257(2004)

[33] Zhongmin Shen, Invent.math.**125**,393(1996)

[34] A. Grigoryan, Bull.American Math.Soc.**36**,135(1999)

[35] H.P. McKean, Comm.Pure Appl.Math.**25**,225(1972)

[36] N. Lohoue and T. Rychener,
Commun.Math. Helv.**57**,445(1982)

[37] J.-P. Anker and L. Li, Geom.func.anal.**9**,1035(1999)

[38] A.Grigoryan and M. Noguchi,
Bull.Lond.Math.Soc.**30**,643(1998)